L.C.M. x H.C.F. = product of numbers ?

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1. For two positive integers a, b :

$$\boxed{a \times b = L.C.M. \times H.C.F.}$$

For example,

$a = 2^3 \times 3 \times 5 = 120$

$b = 2 \times 3^2 = 18$

$L.C.M. = 2^3 \times 3^2 \times 5 = 360$

$H.C.F. = 2 \times 3 = 6$

$\therefore \quad a \times b = 120 \times 18 = 2160$

$\quad L.C.M. \times H.C.F. = 360 \times 6 = 2160$

$\therefore \quad a \times b = L.C.M. \times H.C.F.$

The **proof** is not difficult:

Let the H.C.F. of two numbers a, b be H.C.F.

We then have : $a = H.C.F. \times r$

$b = H.C.F. \times s$

where H.C.F. (r, s) = 1, i.e. r and s have no common factors.

Then

$a \times s = H.C.F. \times r \times s$ ….. **(1)**

$b \times r = H.C.F. \times r \times s$ ….. **(2)**

Now, since r and s have no common factors, we cannot take out a factor greater than 1 in the product $r \times s$. Hence,

$H.C.F. \times r \times s = L.C.M.$ ….. **(3)**

(1) ×**(2)**, $a \times b \times r \times s = (H.C.F.)^2 \times (r \times s)^2$

$\therefore \quad a \times b = (H.C.F. \times r \times s) \times H.C.F.$

$= L.C.M. \times H.C.F.$, by **(3)**

2. Given three positive integers, a, b, c:

$$\boxed{a \times b \times c \neq L.C.M. \times H.C.F.}$$

What is the relation between the product $a \times b \times c$ and L.C.M. and H.C.F. of a, b, c?

Here is an assertion I got through the internet. The original is in Chinese. I make some translation and rearrangements, with the workings shown below. However, the assertion is unfortunately **wrong**. Can you **disprove** it?

Let the H.C.F. of three numbers a, b, c be H.C.F.

We then have : a = H.C.F. × r

b = H.C.F. × s

c = H.C.F. × t

where H.C.F. (r, s,t) = 1, i.e. r, sand t have no common factors.

Then

a × st = H.C.F. × rst ..... **(i)**

b × rt = H.C.F. × rst ..... **(ii)**

c × rs = H.C.F. × rst ..... **(iii)**

Now, since r, s and t have no common factors, we cannot take out a factor greater than 1 in the product rst.

Hence,

H.C.F. × rst = L.C.M. ..... **(iv)**

(i) ×**(ii)** ×**(iii)**, $\mathrm{a} \times \mathrm{b} \times \mathrm{c} \times (\mathrm{rst})^2 = (\mathrm{H.C.F.})^3 \times (\mathrm{rst})^3$

$$\therefore \quad \mathrm{a} \times \mathrm{b} \times \mathrm{c} = (\mathrm{H.C.F.})^3 \times \mathrm{rst}$$
$$= (\mathrm{H.C.F.} \times \mathrm{rst}) \times (\mathrm{H.C.F.})^2$$
$$= \mathrm{L.C.M.} \times (\mathrm{H.C.F.})^2,$$, by **(iv)**.

The best way to **disprove** an assertion is not to find where it makes mistake. (although this is really very good exercise for bright readers to do so.) The best way is to find a **counter-example**.

$$\mathrm{a} = 2 \times 3 \times 5 = 30$$
$$\mathrm{b} = 2^2 \times 3 = 12$$
$$\mathrm{c} = 2$$

Hence $\mathrm{a} \times \mathrm{b} \times \mathrm{c} = 30 \times 12 \times 2 = 720$

$\mathrm{L.C.M.} = 2^2 \times 3 \times 5 = 60$

$\mathrm{H.C.F.} = 2$

$\mathrm{L.C.M.} \times (\mathrm{H.C.F.})^2 = 60 \times 2^2 = 240$

$\therefore \quad \mathrm{a} \times \mathrm{b} \times \mathrm{c} \neq \mathrm{L.C.M.} \times (\mathrm{H.C.F.})^2$

A good counter-example can kill an assertion!

3. L.C.M., H.C.F. and product of 3 numbers

I find the following relation:

Given a, b, c are positive integers,

$$\mathrm{abc} = \frac{\mathrm{LCM(a,b,c)} \times \mathrm{HCF(a,b)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(c,a)}}{\mathrm{HCF(a,b,c)}} \qquad \ldots.(\mathbf{4})$$

Can we have a **proof**? Yes, it is difficult to show, but not impossible.

We guide our proof with the help of a deliberately-made example.

$$\text{Let} \quad \begin{cases} a = 60 \;\;\, = 2 \times 3 \times 5 \qquad\quad \times 2 \\ b = 126 = 2 \times 3 \qquad\quad \times 7 \times 3 \\ c = 70 \;\;\, = 2 \qquad\quad \times 5 \times 7 \end{cases} \qquad \ldots.. \quad \mathbf{(5)}$$

The numbers are written in prime factors, but not in the usual unique representation with prime factors and indices. We take out factors in the following order:

(a) We take out the HCF(a,b,c), in this case is 2.

(b) Then we take out common factors of a and b not included in the H.C.F.(a,b,c).

In this case is 3. Note that H.C.F.(a,b) is 6 and $3 = \frac{6}{2} = \frac{\mathrm{HCF(a,b)}}{\mathrm{HCF(a,b,c)}}$.

(c) Then we take out common factors of b and c not included in the H.C.F.(a,b,c).

In this case is 3. Note that H.C.F.(a,b) is 10 and $5 = \frac{10}{2} = \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}}$.

(d) Then we take out common factors of c and a not included in the H.C.F.(a,b,c).

In this case is 3. Note that H.C.F.(a,b) is 14 and $7 = \frac{14}{2} = \frac{\mathrm{HCF(c,a)}}{\mathrm{HCF(a,b,c)}}$.

(e) Lastly we place all "other factors" in the right-most position. In this case, 2 for a and 3 for b. The "other factors" for a have no common factors with the "other factors" for b or c. Similarly conclusion holds for b's or c's remaining factors.

The given numbers a, b, c in (5) may be more or less complicate, but we can always arrange using the above scheme. Therefore in general:

$$\begin{cases} a = \mathrm{HCF(a,b,c)} \times \frac{\mathrm{HCF(a,b)}}{\mathrm{HCF(a,b,c)}} \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}} \qquad\qquad \times d \\ b = \mathrm{HCF(a,b,c)} \times \frac{\mathrm{HCF(a,b)}}{\mathrm{HCF(a,b,c)}} \qquad\qquad \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}} \times e \\ c = \mathrm{HCF(a,b,c)} \qquad\qquad \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}} \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}} \times f \end{cases} \qquad \ldots.. \quad \mathbf{(6)}$$

where d, e, f are "other factors" mentioned in **(e)** above.

The L.C.M. of the given numerical example **(5)** is $2 \times 3 \times 5 \times 7 \times 2 \times 3 = 1260$.
So the L.C.M. of **(6)** is given by

$$\mathrm{LCM(a,b,c)} = \mathrm{HCF(a,b,c)} \times \frac{\mathrm{HCF(a,b)}}{\mathrm{HCF(a,b,c)}} \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}} \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}} \times \mathrm{def}$$

$$= \frac{\mathrm{HCF(a,b)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(b,c)}}{[\mathrm{HCF(a,b,c)}]^2} \times \mathrm{def} \qquad ..... \quad \mathbf{(7)}$$

The product of the numbers in **(5)** is :

$$\mathrm{abc} = (2 \times 3 \times 5 \times 2)(2 \times 3 \times 7 \times 3)(2 \times 5 \times 7) = (60)(126)(70) = 529200$$

Similarly for **(6),**

$$\mathrm{abc} = [\mathrm{HCF(a,b,c)}]^3 \times \left[\frac{\mathrm{HCF(a,b)}}{\mathrm{HCF(a,b,c)}} \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}} \times \frac{\mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}}\right]^2 \times \mathrm{def}$$

$$= \frac{[\mathrm{HCF(a,b)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(b,c)}]^2}{[\mathrm{HCF(a,b,c)}]^3} \times \mathrm{def}$$

$$= \frac{\mathrm{HCF(a,b)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(b,c)}}{[\mathrm{HCF(a,b,c)}]^2} \times \mathrm{def} \times \frac{\mathrm{HCF(a,b)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}}$$

$$= \mathrm{LCM(a,b,c)} \times \frac{\mathrm{HCF(a,b)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(b,c)}}{\mathrm{HCF(a,b,c)}}$$, by **(7).**

which is the same as

$$\mathrm{abc} = \frac{\mathrm{LCM(a,b,c)} \times \mathrm{HCF(a,b)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(c,a)}}{\mathrm{HCF(a,b,c)}} \qquad ....(\mathbf{4})$$

4. Further on ….

Given a, b, c,d are positive integers,

Let

$$\mathrm{HCF2} = \mathrm{HCF(a,b)} \times \mathrm{HCF(a,c)} \times \mathrm{HCF(a,d)} \times \mathrm{HCF(b,c)} \times \mathrm{HCF(b,d)} \times \mathrm{HCF(c,d)}$$
$$\mathrm{HCF3} = \mathrm{HCF(a,b,c)} \times \mathrm{HCF(a,b,d)} \times \mathrm{HCF(a,c,d)} \times \mathrm{HCF(b,c,d)}$$
$$\mathrm{HCF4} = \mathrm{HCF(a,b,c,d)}$$

$$\mathrm{abcd} = \frac{\mathrm{LCM(a,b,c,d)} \times \mathrm{HCF2} \times \mathrm{HCF4}}{\mathrm{HCF3}} \qquad ....(\mathbf{8})$$

Further on we have

$$\mathrm{abcd} \ldots = \frac{\mathrm{LCM(a,b,c,d,\ldots)} \times \mathrm{HCF2} \times \mathrm{HCF4} \times \mathrm{HCF6} \times \ldots.}{\mathrm{HCF3} \times \mathrm{HCF5} \times \ldots.} \qquad ....(\mathbf{9})$$

where HCF5, HCF6, …. are similarly defined.

5. Exercise

Can you use a spreadsheet such as EXCEL to check the correctness of the above equations **(4), (8)** for the product of 3 or 4 integers? Do you think that equation **(9)** is also correct?